ISSN 2350-1022

International Journal of Recent Research in Mathematics Computer Science and Information Technology
Vol. 11, Issue 2, pp: (14-18), Month: October 2024 – March 2025, Available at: www.paperpublications.org

Matrix Forms of Fractional Euler's Formula and Fractional DeMoivre's Formula

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DOI: https://doi.org/10.5281/zenodo.13986717

Published Date: 24-October-2024

Abstract: **In this paper, based on a new multiplication of fractional analytic functions, we obtain the matrix forms of fractional Euler's formula and fractional DeMoivre's formula. In fact, our results are generalizations of classical calculus results.**

Keywords: **New multiplication, fractional analytic functions, matrix forms, fractional Euler's formula, fractional DeMoivre's formula.**

I. INTRODUCTION

In 1695, the concept of fractional derivative first appeared in a famous letter between L'Hospital and Leibniz. Many great mathematicians have further developed this field. We can mention Euler, Lagrange, Laplace, Fourier, Abel, Liouville, Riemann, Hardy, Littlewood, and Weyl. Fractional calculus has important applications in various fields such as physics, mechanics, electrical engineering, biology, economics, viscoelasticity, control theory, and so on [1-11].

However, different from the traditional calculus, the rule of fractional derivative is not unique, many scholars have given the definitions of fractional derivatives. The common definition is Riemann-Liouville (R-L) fractional derivatives. Other useful definitions include Caputo fractional derivatives, Grunwald-Letnikov (G-L) fractional derivatives, and Jumarie type of R-L fractional derivatives to avoid non-zero fractional derivative of constant function [12-16].

In this paper, based on a new multiplication of fractional analytic functions, we obtain the matrix forms of fractional Euler's formula and fractional DeMoivre's formula. In fact, our results are generalizations of classical calculus results.

II. PRELIMINARIES

Definition 2.1 ([17])**:** If x, x_0, and a_n are real numbers for all n, $x_0 \in (a, b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a, b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x - x_0)^{n\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. In addition, if $f_\alpha: [a, b] \to R$ is continuous on closed interval $[a, b]$ and it is α-fractional analytic at every point in open interval (a, b), then f_α is called an α-fractional analytic function on $[a, b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.2 ([18])**:** If $0 < \alpha \leq 1$. Assume that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional power series at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x - x_0)^{n\alpha}, \quad (1)$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x - x_0)^{n\alpha}. \quad (2)$$

Then

$$f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)$$
$$=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\otimes_\alpha \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}$$
$$=\sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)(x-x_0)^{n\alpha}. \quad (3)$$

Equivalently,

$$f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)$$
$$=\sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes_\alpha n}\otimes_\alpha \sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes_\alpha n}$$
$$=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes_\alpha n}. \quad (4)$$

Definition 2.3 ([19])**:** Let $0<\alpha\le 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}=\sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes n}, \quad (5)$$

$$g_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}=\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes n}. \quad (6)$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha\circ g_\alpha)(x^\alpha)=f_\alpha(g_\alpha(x^\alpha))=\sum_{n=0}^{\infty}\frac{a_n}{n!}(g_\alpha(x^\alpha))^{\otimes n}, \quad (7)$$

and

$$(g_\alpha\circ f_\alpha)(x^\alpha)=g_\alpha(f_\alpha(x^\alpha))=\sum_{n=0}^{\infty}\frac{b_n}{n!}(f_\alpha(x^\alpha))^{\otimes n}. \quad (8)$$

Definition 2.4 ([20])**:** If $0<\alpha\le 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha n}. \quad (9)$$

On the other hand, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \quad (10)$$

and

$$sin_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}. \quad (11)$$

In addition, the α-fractional hyperbolic cosine and hyperbolic sine function are defined as follows:

$$cosh_\alpha(x^\alpha)=\frac{1}{2}[E_\alpha(x^\alpha)+E_\alpha(-x^\alpha)]=\sum_{n=0}^{\infty}\frac{x^{2n\alpha}}{\Gamma(2n\alpha+1)}=\sum_{n=0}^{\infty}\frac{1}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \quad (12)$$

and

$$sinh_\alpha(x^\alpha)=\frac{1}{2}[E_\alpha(x^\alpha)-E_\alpha(-x^\alpha)]=\sum_{n=0}^{\infty}\frac{x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)}=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}, \quad (13)$$

Proposition 2.5 (fractional Euler's formula)**:** *Let* $0<\alpha\le 1$, *then*

$$E_\alpha(ix^\alpha)=cos_\alpha(x^\alpha)+isin_\alpha(x^\alpha). \quad (14)$$

Proposition 2.6 (fractional DeMoivre's formula)**:** *Let* $0 < \alpha \leq 1$, *and* k *be a positive integer, then*

$$[cos_\alpha(x^\alpha) + isin_\alpha(x^\alpha)]^{\otimes_\alpha k} = cos_\alpha(kx^\alpha) + isin_\alpha(kx^\alpha). \quad (15)$$

Definition 2.7: If $0 < \alpha \leq 1$, and A is a matrix. The matrix α-fractional exponential function is defined by

$$E_\alpha(Ax^\alpha) = \sum_{n=0}^{\infty} A^n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(A\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha n}. \quad (16)$$

III. MAIN RESULTS

In this section, we provide the matrix forms of fractional Euler's formula and fractional DeMoivre's formula.

Theorem 3.1 (matrix form of fractional Euler's formula)**:** *If* $0 < \alpha \leq 1$, E *is the unit matrix and* I, J *are matrices such that* $I^2 = -E, J^2 = E$. *Then*

$$E_\alpha(Ix^\alpha) = Ecos_\alpha(x^\alpha) + Isin_\alpha(x^\alpha) \quad , \quad (17)$$

and

$$E_\alpha(Jx^\alpha) = Ecosh_\alpha(x^\alpha) + Jsinh_\alpha(x^\alpha) \quad , \quad (18)$$

Proof $E_\alpha(Ix^\alpha)$

$$= \sum_{n=0}^{\infty} I^n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)}$$

$$= E + I\frac{x^\alpha}{\Gamma(\alpha+1)} + I^2\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + I^3\frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + I^4\frac{x^{4\alpha}}{\Gamma(4\alpha+1)} + \cdots$$

$$= E + I\frac{x^\alpha}{\Gamma(\alpha+1)} - E\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} - I\frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + E\frac{x^{4\alpha}}{\Gamma(4\alpha+1)} - \cdots$$

$$= E\left[1 - \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{4\alpha}}{\Gamma(4\alpha+1)} - \cdots\right] + I\left[\frac{x^\alpha}{\Gamma(\alpha+1)} - \frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots\right]$$

$$= Ecos_\alpha(x^\alpha) + Isin_\alpha(x^\alpha).$$

On the other hand,

$$E_\alpha(Jx^\alpha)$$

$$= \sum_{n=0}^{\infty} J^n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)}$$

$$= E + J\frac{x^\alpha}{\Gamma(\alpha+1)} + J^2\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + J^3\frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + J^4\frac{x^{4\alpha}}{\Gamma(4\alpha+1)} + \cdots$$

$$= E + J\frac{x^\alpha}{\Gamma(\alpha+1)} + E\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + J\frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + E\frac{x^{4\alpha}}{\Gamma(4\alpha+1)} + \cdots$$

$$= E\left[1 + \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^{4\alpha}}{\Gamma(4\alpha+1)} + \cdots\right] + J\left[\frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{x^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots\right]$$

$$= Ecosh_\alpha(x^\alpha) + Jsinh_\alpha(x^\alpha).$$

q.e.d.

Theorem 3.2 (matrix form of DeMoivre's formula)**:** *If* $0 < \alpha \leq 1$, n *is an integer,* E *is the unit matrix and* I, J *are matrices such that* $I^2 = -E, J^2 = E$. *Then*

$$[Ecos_\alpha(x^\alpha) + Isin_\alpha(x^\alpha)]^{\otimes_\alpha n} = Ecos_\alpha(nx^\alpha) + Isin_\alpha(nx^\alpha), \quad (19)$$

and

$$[Ecosh_\alpha(x^\alpha) + Jsinh_\alpha(x^\alpha)]^{\otimes_\alpha n} = Ecosh_\alpha(nx^\alpha) + Jsinh_\alpha(nx^\alpha). \quad (20)$$

Proof By matrix form of fractional Euler's formula,

$$[Ecos_\alpha(x^\alpha) + Isin_\alpha(x^\alpha)]^{\otimes_\alpha n}$$

$$= [E_\alpha(Ix^\alpha)]^{\otimes_\alpha n}$$
$$= E_\alpha(Inx^\alpha)$$
$$= Ecos_\alpha(nx^\alpha) + Isin_\alpha(nx^\alpha)\ .$$

And

$$[Ecosh_\alpha(x^\alpha) + Jsinh_\alpha(x^\alpha)]^{\otimes_\alpha n}$$
$$= [\ E_\alpha(Jx^\alpha)]^{\otimes_\alpha n}$$
$$= E_\alpha(Jnx^\alpha)$$
$$= Ecosh_\alpha(nx^\alpha) + Jsinh_\alpha(nx^\alpha)\ .$$

q.e.d.

IV. CONCLUSION

In this paper, based on a new multiplication of fractional analytic functions, we obtain the matrix forms of fractional Euler’s formula and fractional DeMoivre’s formula. In addition, our results are generalizations of ordinary calculus results. In the future, we will continue to use the new multiplication of fractional analytic functions to solve the problems in fractional differential equations and applied mathematics.

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ISSN 2350-1022

International Journal of Recent Research in Mathematics Computer Science and Information Technology
Vol. 11, Issue 2, pp: (14-18), Month: October 2024 – March 2025, Available at: www.paperpublications.org

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